

# Introduction to causal sets: an alternate view of spacetime structure

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**Abstract.** This paper provides a thorough introduction to the causal set hypothesis aimed at students, and other interested persons, with some knowledge of general relativity and nonrelativistic quantum mechanics. I elucidate the arguments for why the causal set structure might be the appropriate structure for a theory of quantum gravity. The logical and formal development of a causal set theory as well as a few illuminating examples are also provided.

## I. INTRODUCTION

When studying general relativity, students often find that two of the most compelling topics, cosmology and black holes, lead directly to the need for a theory of quantum gravity. However, not much is said about quantum gravity at this level. Those who search for more information will find that most discussions center around the two best known approaches: canonical quantization [1] and superstring theories [2]. This paper seeks to introduce the problem of quantum gravity in the context of a third view, causal sets, which has emerged as an important concept in the pursuit of quantum gravity.

The causal set idea is an hypothesis for the structure of spacetime. This structure is expected to become apparent for extremely tiny lengths and extremely short times. This hypothesis, in its current form, has grown out of an attempt to find an appropriate structure for a physical theory of quantum gravity. There is a long tradition of the importance of causality in relativity. Many of the issues faced when confronting the problem of quantum gravity bring considerations of time and causality to the forefront.

There are many approaches to quantum gravity. Usually, these approaches go through cycles of rapid progress, during which times an approach will appear very promising, followed by (sometimes long) periods of slow, or no, progress. The causal set approach has gone through these cycles as well, although to a lesser extent than some, with early work by Finkelstein [3], Myrheim [4], 't Hooft [5], and Sorkin [6]. The recent upswing of interest in causal sets was ignited by a paper written in the late 1980s [7]. Since the hope is that causal sets will lead to a working model for quantum gravity, it seems appropriate to begin by describing the problem of quantum gravity in general. The basic ideas behind the causal set approach and some of the progress that has been forged in recent years will be discussed in sections III - VI.

## II. THE PROBLEM OF QUANTUM GRAVITY

### A. What is quantum gravity?

The question of what one means by “quantum gravity” is not a simple question to answer for the obvious reason that we do not yet have a complete understanding of quantum gravity. Hence, the answers to this question are both short and long and perhaps as numerous as the number of approaches attempting to solve the problem. Most physicists agree that by “gravity” we mean Einstein’s theory of general relativity (and possibly a few modified versions of it). General relativity most popularly interprets gravitation as a result of the geometrical structure of spacetime. The geometrical interpretation fits because the theory is formally cast in terms of metrical structure  $g_{\mu\nu}$  on a manifold  $M$ .

There is somewhat less agreement on the meaning of “quantum.” At first glance, it seems odd that there would be less agreement on the aspect with which we have much more experience. On the other hand, however, the fact that we have only been able to perform weak-field experimental tests of general relativity leaves us with much less information to debate. Our experience with quantum mechanics tells us that the deviations from classical physics it describes are important when dealing with size scales on the order of magnitude of an atom and smaller. Is there a natural size scale at which we expect the predictions of general relativity to be inaccurate requiring a new more fundamental theory?

The scale at which theories become important is set by the values of the fundamental parameters related to the processes being described. For example, the speed of light  $c$  is the fundamental constant that determines the velocity scale for which relativistic effects (special relativity) are appreciable. Likewise, Planck’s constant  $\hbar$ , among others, sets the scale for systems that must be described by quantum mechanics. The fundamental constants that are relevant to a theory of quantum gravity are the speed of light, Planck’s constant, and the universal gravitation constant  $G$ . These three quantities combine to form the length and time scales at which classical general relativity break down:

$$\ell_P = (G\hbar/c^3)^{1/2} \sim 10^{-35} \text{ m} \quad (1)$$

$$t_P = \ell_P/c \sim 10^{-44} \text{ s},$$

where  $\ell_P$  is called the Planck length and  $t_P$  is called the Planck time.

Size, however, is only one part of what makes a theory “quantum.” Consider, once again, the atom. If we

dig deeper than just size and ask why quantum effects are important for atoms, the answer is that a relatively small number of states are occupied (or excited). This fact is more commonly stated in reverse as a correspondence principle requiring that quantum mechanics merge with classical mechanics in the limit of large quantum numbers, that is, a large number of occupied states. It is this latter point that truly characterizes quantum behavior. A quantum theory must therefore enumerate and describe the states of a system in such a way that the known classical behavior emerges for large numbers of states.

What, then, do we mean by “quantum gravity?” In this paper, my working definition is that

*quantum gravity is a theory that describes the structure of spacetime and the effects of spacetime structure down to sub-Planckian scales for systems containing any number of occupied states.*

In the above definition, the “effects of spacetime structure” include not only the phenomenon of gravitational attraction, but also any implications that the spacetime dynamics will have for other interactions that take place within this structure.

## B. Why do we need quantum gravity?

### 1. The Einstein field equations

The content of the Einstein field equations of general relativity,

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \quad (2)$$

suggests the need for a quantum mechanical interpretation of gravity [8]. Here  $G_{\mu\nu}$  is the Einstein curvature tensor representing the curvature of space-time,  $T_{\mu\nu}$  is the energy-momentum tensor representing the source of gravitation, while  $\kappa$  is just a coupling constant between the two. The energy-momentum content of spacetime is already known to be a quantum operator from other fundamental theories such as quantum electrodynamics (QED). We have confidence in the reliability of this interpretation because, despite the fact that QED may have flaws (discussed below), it has led to extremely accurate agreement between theory and experiment [9]. Since energy-momentum is a quantum operator whose macroscopic version is intimately related to macroscopic space-time structure, it seems a good working hypothesis that its quantum mechanical version should correspond to a structure of space and time appropriate in the quantum mechanical regime.

### 2. Black holes

A *black hole* is the final stage in the evolution of massive stars. Black holes are formed when the nuclear energy source at the core of a star is exhausted. Once the nuclear fuel has run out, the star collapses. If the remaining mass of the star is sufficiently high, no known force can halt the collapse. General relativity predicts that the stellar mass will collapse to a state of zero extent and infinite density – a *singularity*. In this singular state there is no spatial extent, time has no meaning, and the ability to extract any physical information is lost. This prediction may be a message which tells us that a quantum theory of gravity is needed if we are to truly understand the inner workings of black holes.

While it may be obvious that processes deep within black holes must be treated in the framework of quantum gravity, it is less obvious that processes well away from the singularity not only require quantum gravity, but may also provide important clues to the form a theory of quantum gravity should take. In 1975 Hawking [10] showed that black holes radiate thermally with a black-body spectrum. This finding, together with a previous conjecture that the area of a black hole’s event horizon can be interpreted as its entropy [11], has shown that the laws of black hole mechanics are identical to the laws of thermodynamics. This equivalence only comes about if we accept the *identification* of the area of the black hole (actually 1/4 of it) as its entropy. In traditional thermodynamics the concept of entropy is best understood in terms of discrete quantum states; not surprisingly, attempts to better understand the reasons for this area identification using classical gravity fail. It is widely expected that only a quantum mechanical approach will produce a satisfactory explanation [12]. For this reason, black hole entropy is an important topic for most approaches to quantum gravity [13].

### 3. The early universe

One of the many triumphs of relativistic cosmology is the explanation of the observed redshift of distant galaxies as an expansion of the universe. However, the universal expansion extrapolates backward to an early universe that is infinitesimally small and infinitely dense – the big bang singularity. Here then, is another situation in which general relativity predicts something it is not equipped to describe. It is fully expected that events near the singularity were dominated by quantum mechanical influences both *of* and *on* spacetime which necessarily affects the subsequent evolution of the universe. Presently, cosmological implications of the early universe are studied with the techniques of quantum cosmology [14] which is the quantum mechanics of classical cosmological models. It has been pointed out that quantum cosmology cannot be trusted except in very specialized cases [15]. While

we have good theories for doing quantum mechanics on background spacetimes, the early universe problem requires a theory for the quantum mechanics of spacetime itself – a theory of quantum gravity.

#### *4. Unification*

Throughout the history of physics, great strides have been made through the unification of seemingly different aspects of nature. One of the most prominent examples is Maxwell's unification of the laws of electricity and magnetism. Einstein's theory of special relativity amounts to a unification of Maxwell's electromagnetism and Newton's mechanics showing Newton's laws to be merely a “low speed” approximation to a more accurate relativistic dynamics. Following relativity theory, quantum mechanics was born. Soon thereafter, Dirac unified quantum mechanics and special relativity giving rise to modern quantum field theory.

With the above successful unifications behind us, we are left with the present situation of having several fundamental forces known as the strong, weak, and electromagnetic interactions as well as gravitation. Given the benefits that we have reaped from past unifications it seems natural that the search for deeper insight through unification should continue. The recent success of the electroweak theory has confirmed the value of this search. There are now some seemingly consistent models for the unification of the strong and electroweak theories. The very fact that these interactions can be mathematically unified in a manner consistent with macroscopic observations suggests that a truly physical unification exists.

Gravity is the only fundamental force yet to have a consistent quantum mechanical theory. It is widely believed that until such a quantum mechanical description of gravity is attained, placing our understanding of gravity on the same level as that of the other interactions, true unification of gravity with the other forces will not be possible [8].

### C. The incompatibility between general relativity and quantum mechanics

For all of the interactions except gravity, our present theoretical understanding of physics is such that systems interact and evolve within a background spacetime structure. This background structure serves to tell us how to measure distances and times. In general relativity it is the spacetime structure itself that we must determine. This spacetime structure then, acts both as the background structure for gravitational interactions and as the dynamical phenomenon giving rise to this interaction. In general relativity the structure of spacetime is determined by the Einstein field equations (2). These field equations are, however, purely classical in that they

do not meet the requirements of a quantum theory as discussed in section II.A above. The breakdown of general relativity near the singularity of a black hole, or more accurately, the prediction of a singularity inside of a black hole, is just one of many examples. Given that general relativity was formulated prior to quantum mechanics, the fact that it does not meet quantum mechanical requirements is not surprising.

The dual role of the metric tensor makes formulating a theory of quantum gravity very different from the formulations of the other interactions. In quantum gravity we must determine the spacetime structure that acts as background to the classical structure of space and time that we have used to understand all other phenomena. Furthermore, this ultimate background to classical spacetime structure must also be dynamic because it is this dynamics that will describe quantum gravity just as the dynamics of classical spacetime describes general relativity. This latter point is the key reason for the incompatibility between general relativity and quantum mechanics. All of our successful experience is with quantum dynamics on a spacetime structure, but we have had very little success handling the quantum dynamics of spacetime structure.

This incompatibility challenges some of the most fundamental concepts in physics. In field theory, we take as the source of the field some distribution  $T_{\mu\nu}$  (of charge, matter, energy, etc.). However, the concept of a spatial distribution of charge, for example, has no meaning apart from the knowledge of the spatial structure. Without the rules for how to measure relative positions we cannot define the spatial distribution of anything. In quantum mechanics, Schrödinger's equation describes the time evolution of the wave function  $\Psi(\mathbf{r}, t)$ . The concept of time evolution, however, requires an existing knowledge of temporal flow which comes from the spacetime structure. As a final example, consider the concept of interaction. We generally think of interactions in a manner intimately connected with causality: an interaction precedes and causes an effect. Causality, however, is a concept that can only be defined once the structure of spacetime is known.

### III. THE CAUSAL SET HYPOTHESIS

The above discussion implies the need for a spacetime structure that will underpin the classical spacetime structure of general relativity. The causal set hypothesis proposes such a structure. Causal sets are based on two primary concepts: the discreteness of spacetime and the importance of the causal structure. Below I discuss these two founding concepts in more detail.

### A. Spacetime is discrete

The causal set hypothesis assumes that the structure of spacetime is discrete rather than the continuous structure that physics currently employs. Discrete means that lengths in three-dimensional space are built up out of a finite number of elementary lengths  $\ell_e$  which represents the smallest allowable length in nature and the flow of time occurs in a series of individual “ticks” of duration  $t_e$  which represents the shortest allowable time interval.

The idea that something which appears continuous is actually discrete is very common in physics and everyday life. Any bulk piece of matter is made up of individual atoms so tightly packed that the object appears continuous to the naked eye. Likewise, any motion picture is constructed of a series of snapshots so rapidly paced that the movie appears to flow continuously.

Why hold a similar view of spacetime? There are many arguments for a discrete structure. The most familiar ones are related to electrodynamics. Here, I will first try to motivate the idea of discreteness by considering the electromagnetic spectrum. The electromagnetic spectrum gives us the range for the frequencies and wavelengths of electromagnetic radiation, or, photons. Of the many interesting aspects of this spectrum, here let’s focus on that fact that it is a continuous spectrum of infinite extent. Current theory predicts that the allowed frequencies of photons extend continuously from zero to infinity. The relation  $E = h\nu$  implies that a photon of arbitrarily large frequency has arbitrarily large energy. The local conservation of energy suggests that such infinite energy photons should not exist. If one adopts the (somewhat controversial) view that what cannot physically exist should not be predicted by theory, there ought to be a natural cutoff of the electromagnetic spectrum corresponding to a maximum allowed frequency. Since the frequency of a photon is the inverse of its period, a discrete temporal structure provides a natural cutoff in that the minimum time interval  $t_e$  implies a maximum frequency  $\nu_{\max} = 1/t_e$ .

Because of relativity any argument for discrete time is also an argument for discrete space. Nevertheless, a similar argument for discrete space can be given in terms of wavelength. The electromagnetic spectrum, being continuous, allows arbitrarily small wavelengths. The de-Broglie relation  $p = h/\lambda$  implies that a wavelength arbitrarily close to zero corresponds to a photon of arbitrarily large momentum. The local conservation of momentum suggests that photons of infinite momentum should not exist. The minimum length implied by a discrete spacetime structure provides a natural cutoff for the wavelength  $\lambda_{\min} = \ell_e$ .

In terms of QED, this problem can be seen in the fact that the infinite perturbation series requires the existence of all the photons in the electromagnetic spectrum. In this sense QED predicts the existence of these photons of infinite energy-momentum. However, it is widely be-

lieved that the perturbation series diverges. This divergence is generally overlooked because QED is only a partial theory and not a complete theory of elementary interactions (see Ch. 1 of Ref. 1). Therefore, we use QED under the assumption that it is accurate for the phenomenon it was created to describe and that some aspect of a more fundamental theory will eventually solve its divergence problem. The causal set idea proposes that the aspect of more fundamental theory that will naturally solve this divergence problem in QED is a discrete structure of spacetime.

### B. Causal structure contains geometric information

Spacetime consists of events  $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$ , that is, points in space at various times. At some events physical processes take place. Processes that occur at one event can only be influenced by those occurring at another event if it is possible for a photon to reach the latter event from the earlier one. To capture the essence of what one means by “causal structure,” consider the example of the flat Minkowski space of special relativity. In flat spacetime, two events are said to be causally connected and their spacetime separation  $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$  is called timelike if it is positive (using a  $+++ -$  signature) and null if it is zero. Two events are not causally connected if it is not possible for a photon from one event to arrive at the other; these events cannot influence each other and in such cases  $ds^2$  is negative and referred to as spacelike. When we speak of the causal structure of a spacetime, we mean the knowledge of which events are causally connected to which other events. For a more general discussion of causal structure see reference [16].

It is now well established that the causal structure of spacetime alone determines almost all of the information needed to specify the metric [17, 18] and therefore the gravitational field tensor. The causal structure determines the metric up to an overall multiplicative function called a conformal factor. We say that two metrics  $g_{\mu\nu}$  and  $\tilde{g}_{\mu\nu}$  are conformally equivalent if  $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ , where  $\Omega$  is a smooth positive function. Since all conformally equivalent spacetimes have the same causal structure [19], the causal structure itself nearly specifies the metric.

### C. Causal sets

Lacking the conformal factor from the causal structure essentially means that we lack the sense of scale which allows for quantitative measures of lengths and volumes in spacetime. However, if spacetime is discrete, the volume of a region can be determined by a procedure almost as simple as counting the number of events within that region. Therefore, if nature endows us with discrete

spacetime and an arrow for time (the causal structure), we have, in principle, enough information to build complete spacetime metric tensors for general relativity. This combination of discreteness and causal structure leads directly to the idea of a causal set as the fundamental structure of spacetime.

A causal set may be defined as a set of events for which there is an order relation  $\prec$  obeying four properties:

1. *transitivity*: if  $x \prec y$  and  $y \prec z$  then  $x \prec z$ ;
2. *non-circularity*: if  $x \prec y$  and  $y \prec x$  then  $x = y$ ;
3. *finitarity*: the number of events lying between any two fixed events is finite;
4. *reflexivity*:  $x \prec x$  for any event in the causal set.

The first two properties say that this ordered set is really a partially ordered set, or *poset* for short. Specifically, non-circularity amounts to the exclusion of closed timelike curves more commonly known as time machines [20]. The finitarity of the set insures that the set is discrete. The reflexivity requirement is present as a convenience to eliminate the ambiguity of how an event relates to itself. In the present context of using a poset to represent spacetime, reflexivity seems reasonable in that the spacetime separation between an event and itself cannot be negative requiring an event to be causally connected to itself. We can combine these statements to give the following definition:

*A causal set is a locally finite, partially ordered set.*

#### IV. THE DEVELOPMENT OF CAUSAL SET THEORY

With the conceptual foundation of causal sets clearly laid, let us now turn to the issue of developing a physical and mathematical formalism which I loosely refer to as *causal set theory*. The development of causal set theory is still far from complete. In fact, it is even less developed than some of the other approaches to quantum gravity such as superstring theory and canonical quantization mentioned previously. For this reason, quantum gravity by any approach is an excellent theatre for fine tuning ideas about how theory construction should proceed from founding observations, hypotheses, and assumptions. Hence, to better understand current thought on how the development of causal set theory should proceed, I will first discuss some of the ideas on theory development in general that have influenced causal set research.

##### A. Taketani stages

In 1971 Mituo Taketani used Newtonian mechanics as a prototype to illustrate his ideas on the development of physical theories [21]. According to Taketani, physical theories are developed in three stages that he referred to as the phenomenological, substantialistic, and essentialistic stages.

The phenomenological stage is where the initial observations occur that place the existence and knowledge of the new phenomenon (or “substance”) on firm standing. In Ref. [21] this stage in the development of Newtonian mechanics is associated with the work of Tycho Brahe who observed the motions of the planets with unprecedented accuracy.

Within the substantialistic stage, rules that describe the new substance are discovered; that is, we come to recognize “substantial structure” in the new phenomenon. These rules would then play an important role in helping to shape the final understanding. For Newtonian mechanics, Taketani associated this stage with the work of Johannes Kepler who provided the well known three laws of planetary motion.

In the essentialistic stage, “the knowledge penetrates into the essence” of the new phenomenon. This is the final stage when the full theory of this new substance is known within appropriate limits of validity. Of course, the work of Issac Newton himself represents this stage.

Even though Taketani used Newtonian mechanics, there are many examples to which his ideas apply. Sakata used Taketani’s philosophy to discuss the development of quantum mechanics [22]. Similarly, the development of electromagnetic theory falls neatly into Taketani’s framework. The phenomenological stage of electromagnetism could be associated with the work of Benjamin Franklin and William Gilbert. The substantialistic stage is nicely represented by the work of Michael Faraday and Hans Oersted. The essentialistic stage is then represented by James Maxwell’s completion of his famous equations.

Taketani realized that his three stages will not always apply identically to the development of all physical ideas. Since we currently know of no observed phenomena whose explanation clearly requires a complete theory of quantum gravity, it is clear that the problem of quantum gravity is not based on experimental observations. As a result of this fact, the development of causal set theory largely skips the phenomenological stage. Therefore, think of the development of causal sets in a two-step process corresponding roughly to Taketani’s substantialistic and essentialistic stages. As a matter of terminology, note that the substantialistic stage, in which phenomenology is described, plays the role of kinematics in Newtonian mechanics, while in the essentialistic stage the full dynamics is developed. Consequentially, I will refer to the two processes in the development of causal set theory as “kinematics” and “dynamics.”

## B. Causal set kinematics

The kinematic stage concerns gaining familiarity with and further developing the mathematics needed to describe causal sets. This mathematics primarily falls under the combinatorial mathematics of partial orders [23]. These techniques are not part of traditional physics training and have, therefore, not been widely used to analyze physical problems. Moreover, research in this branch of mathematics has been performed largely by pure mathematicians; the problems they have chosen to tackle are generally not those that are of most interest to physicists. What we need from the kinematic stage are the mathematical techniques for how to extract the geometrical information from the causal order (i.e., working out the correspondence between order and geometry) and how to do the counting of causal set elements that will allow us to determine spacetime volumes.

For an important, specific example of where causal set kinematics is needed, consider the correspondence principle between spacetime as a causal set and macroscopic spacetime. General relativity tells us that spacetime is a four-dimensional Lorentzian manifold. If causal sets comprise the true structure of spacetime they must produce a four-dimensional Lorentzian manifold in macroscopic limits (such as a large number of causal set elements). The mathematics of how we can see the manifold within the causal set is a kinematic issue that must be addressed.

On the natural length scale of the causal set one does not expect to see anything like a manifold. Trying to see a manifold on this scale is like trying to read this article under a magnification that resolves the individual dots of ink making up the letters. To discern the structure of these dots we look upon them at a significantly different scale than the size scale of the dots. Similarly, we need a mathematical change-of-scale in order to extract the manifold structure from the causal set. This change-of-scale is called *coarse-graining*.

Some insight into this issue can be gained by looking at the reverse problem of forming a causal set from a given metric manifold  $(M, g_{\mu\nu})$ . This is achieved by randomly sprinkling points into  $M$ . The order relation of this set of points is then determined by the light-cone structure of  $g_{\mu\nu}$ . Since we need to ensure that every region of the spacetime is appropriately sampled, that is, that highly curved regions are represented equally well as nearly flat regions are, the sprinkling is carried out via a Poisson process [24] such that the number of points  $N$  sprinkled into any region of volume  $V$  is directly proportional to  $V$ . Using a two-dimensional Minkowski spacetime, Fig. 1 provides a picture of such a sprinkling at unit density  $\rho = N/V = 1$ .

Since the causal sets generated by random sprinklings are only expected to be coarse-grained versions of the fundamental causal set, their length and time scales are not expected to be the fundamental length and time of nature. Nevertheless, these studies of random sprinklings

are important because the founding ideas behind causal sets in Sec. III suggests that *a manifold  $(M, g_{\mu\nu})$  emerges from the causal set  $C$  if and only if an appropriately coarse-grained version of  $C$  can be produced by a unit density sprinkling of points into  $M$*  [25]. This shows us that an important problem in the development of causal set kinematics is to determine how to appropriately form a coarse-graining of a causal set.

## C. Causal set dynamics

The final stage in the development of causal set theory is the stage in which we come to understand the full dynamics of causal sets. In this stage we devise a formalism for how to obtain physical information from the behavior of the causal set and how this behavior governs our sense of space and time. Here we require something that might be considered a quantum mechanical analog to the Einstein field equations (2). Since our present framework for physical theories is based on a spacetime continuum, our experience is of limited use to us in this effort. Despite this limitation, one commonly used approach stands out as the best candidate for a dynamical framework for causal sets. This method is most commonly known as the path-integral formulation of quantum mechanics [26].

This path-integral technique seems best suited to causal sets because at its core conception (a) it is a spacetime approach in that it deals directly (rather than indirectly) with events; that is, we propagate a system from one event configuration to another; and (b) it works on a discrete spacetime structure. As currently practiced, the path-integral approach determines the propagator  $U(a^\mu, b^\nu)$  by taking all paths between the events  $a^\mu$  and  $b^\nu$  in a discretized time and summing over these paths using an amplitude function  $\exp(iS/\hbar)$ :

$$U(a^\mu, b^\nu) \sim \sum_{\text{all paths}} \exp(iS/\hbar), \quad (3)$$

where  $S$  is the action for a given path. Continuous spacetime enters in at two places. In a continuum there are an infinite number of paths between two events, “all paths” are generated by integrating over all intermediate points between the two events; this is the “integral” part of path-integration. Since each of these paths were discretized into a finite number of points  $N$ , the second place where continuous spacetime is recovered is to take the limit  $N \rightarrow \infty$ .

In a discrete setting the number of paths and the number of points along the paths are truly finite. Hence, the final limit as well as the integration to generate all paths are not performed. Since causal sets would not require integration, calling this the “path-integral formulation” seems inappropriate. This method essentially says that the properties of a system in a given event configuration depend on a sum over all the possible paths throughout the history of this system. Therefore, the alternative

name for this technique, the *sum-over-histories* approach, is better suited for causal sets. The word “histories” is particularly appropriate because, as mentioned earlier, we take the arrow of time to be fundamental.

There are several key issues that must be resolved before a sum-over-histories formulation of causal sets can be completed. One such issue involves the need to identify an amplitude function for causal sets analogous to the role played by  $\exp(iS/\hbar)$ . Secondly, the required formulation must do more than just propagate the system because the entire dynamics must come from this formalism. The procedure outlined above is presently inadequate for these purposes; a modified, or better, generalized sum-over-histories method must be developed.

Perhaps the most significant advance along the dynamical front is the recent development by Rideout and Sorkin of a general *classical* dynamics for causal sets [27]. In this model, causal sets are grown sequentially, one element at a time, under the governance of reasonable physical requirements for causality and discrete general covariance. When a new element is introduced, in going from an  $n$ -element causal set to an  $(n+1)$ -element causal set, it is associated with a classical probability  $q_n$  of being unrelated to any existing element according to

$$\frac{1}{q_n} = \sum_{k=0}^n \binom{n}{k} t_k.$$

The primary restriction is that the  $t_k \geq 0$ ; hence, there is a lot of freedom with which different models can be explored. This framework has the potential to teach us much about the needed mathematical formalism for causal sets, the effects of certain physical conditions, and the classical limit of the eventual quantum dynamics for causal sets.

## V. ILLUSTRATIVE EXAMPLES

To illustrate some of the points discussed above I present the 72 element causal set shown in Fig. 2. The black dots represent the elements of the causal set. The graphical form in which this causal set is shown is known as a *spacetime-Hasse* diagram. The term “Hasse diagram” is borrowed from the mathematical literature on posets [23]. Figure 2 is also a spacetime diagram in the usual sense. The solid lines in the figure are *causal links*, i.e., lines are only drawn between events that are causally related; however, for clarity only those relations that are not implied by transitivity (the links) are explicitly shown. The causal set shown has 15 time steps as enumerated along the right side of the figure. Hence, the first time step at the bottom shows 7 “simultaneous” events.

### A. Volume

As discussed in Sec. III, the causal set hypothesis is partially founded on the fact that the causal structure of spacetime contains all of the geometric information needed to specify the metric tensor up to a conformal factor which prevents us from determining volumes. One example which shows how, in principle, volumes might be extracted from the causal set has been discussed by Bombelli [28]. Gerard ’t Hooft has shown [5] that, in Minkowski space, the volume  $V$  of spacetime bounded by two causally connected events  $a$  and  $b$  is given by

$$V = \frac{\pi \tau_{ab}^4}{24}, \quad (4)$$

where  $\tau_{ab}$  is the proper time between events  $a$  and  $b$ . We can apply this expression to causal sets by relating  $\tau_{ab}$  to the number of links in the longest path between  $a$  and  $b$ . The volume  $V$  is then identified with the number of elements in this region of spacetime.

Spacetime is dynamic, however. The above procedure of counting the number of links between two events is subject to (perhaps large) statistical fluctuations. Therefore, while it is believed that the expected proper time  $\langle \tau \rangle$  should be proportional to the number of links [29], the precise relationship between them is yet unknown. Attempts to numerically determine this relationship via computer simulations remain inconclusive [28].

### B. Coarse-graining

While the labeling of Fig. 2 clearly suggests that it is a two-dimensional example of a causal set, note that our physical sense of dimensionality (given us by relativity) is intimately related to the manifold concept. Although the causal set in Fig. 2 looks suspiciously regular it may not be immediately obvious whether or not this set can be embedded into any physically viable two-dimensional spacetime.

As an example of one form of coarse-graining, we can look at this causal set on a time scale twice as long as its natural scale. Figure 3a is a coarse-grained version of the causal set in Fig. 2 for which only even time steps are shown and Fig. 3b is a subset of Fig. 2 showing events that only occur at odd time steps. In both cases we find causal sets that clearly can be embedded into two-dimensional Minkowski space. In a realistic situation this fact would suggest that the fundamental causal set just might represent a physically discrete spacetime.

### C. Dynamics

As stated above, a sum-over-histories dynamical law for causal sets requires the identification of an amplitude

function. As an example, one could start by considering an amplitude modeled after the familiar amplitude of the continuum path-integral formulation in Eq. (3), i.e.,  $\exp(i\beta R)$ . Here,  $\beta$  plays the role of  $1/\hbar$  and  $R$  plays the role of the action  $S$ . In quantum field theory the oscillatory nature of this amplitude causes problems that are sometimes bypassed by performing a continuation from real to imaginary time (often referred to as a Wick rotation). Similarly, it is convenient here to consider the case  $\beta \rightarrow i\beta$  giving an amplitude

$$A = \exp(-\beta R), \quad (5)$$

where we can take  $R$ , for example, to be the total number of links in the causal set.

This model is interesting because the amplitude  $A$ , which acts as a weight in the sum-over-causal sets, has the form of a Boltzmann factor  $\exp(-E/kT)$ . The mathematical structure of the causal set dynamics then becomes very similar to that of statistical mechanics. Studies of the statistical mechanics of certain partially ordered sets have been performed [30]. In the thermodynamic limit, these studies exhibit phase transitions corresponding to successively increasing numbers of layers of the lattice causing the poset to appear more and more continuous. In this analogy, the thermodynamic limit corresponds to one macroscopic limit of causal set theory in which the number of causal set elements goes to infinity. Such results, therefore, are somewhat suggestive that an appropriate choice of amplitude might indeed lead to the expected kind of continuum limit.

Another, more detailed, example of a quantum dynamics for causal sets that exhibits the kind of interference effects that are absent from the classical dynamics mentioned previously can be found in Ref. [31].

## VI. CONCLUDING REMARKS

In this paper we have tried to communicate the primary motivation and key ideas behind the causal set hypothesis. Causal sets has emerged as an important approach to quantum gravity having been found to impact other approaches such as the spin network formalism [32].

Adding to the importance of the causal set approach is the fact that it led Sorkin to predict a non-zero cosmological constant nearly a decade ago [25]. In light of recent findings in the astrophysics community [33, 34], this result perhaps marks the only prediction to come out of quantum gravity research that might be testable in the foreseeable future.

Before a final causal set theory can be constructed, much work remains. Studies of random sprinklings in both flat and curved spacetimes, the mathematics of partial orders, and the behavior of fields that sit on a discrete substructure are just a few areas of needed investigation. Enough progress on causal sets has been made, however, to establish the causal set hypothesis as a very promising

branch of quantum gravity research. Those with further interest can find more detailed discussion of causal sets in Refs. [35, 36].

## VII. ACKNOWLEDGMENTS

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## VIII. FIGURE CAPTIONS

**Figure 1.** A causal set formed from a unit density sprinkling of points in two-dimensional Minkowski space.

**Figure 2.** A spacetime-Hasse diagram of a two-dimensional causal set. The dots represent the 72 events in this set and the lines are causal links between events. This causal set has 15 time steps as enumerated along the right-hand-side of the figure.

**Figure 3.** Coarse-grainings of the causal set in Fig. 2 formed by doubling the time scale. (a) The subset formed by the odd time steps only. (b) The subset formed by the even time steps only. Both coarse-grainings are more clearly embeddable in two-dimensional Minkowski space than the full poset in Fig. 2.

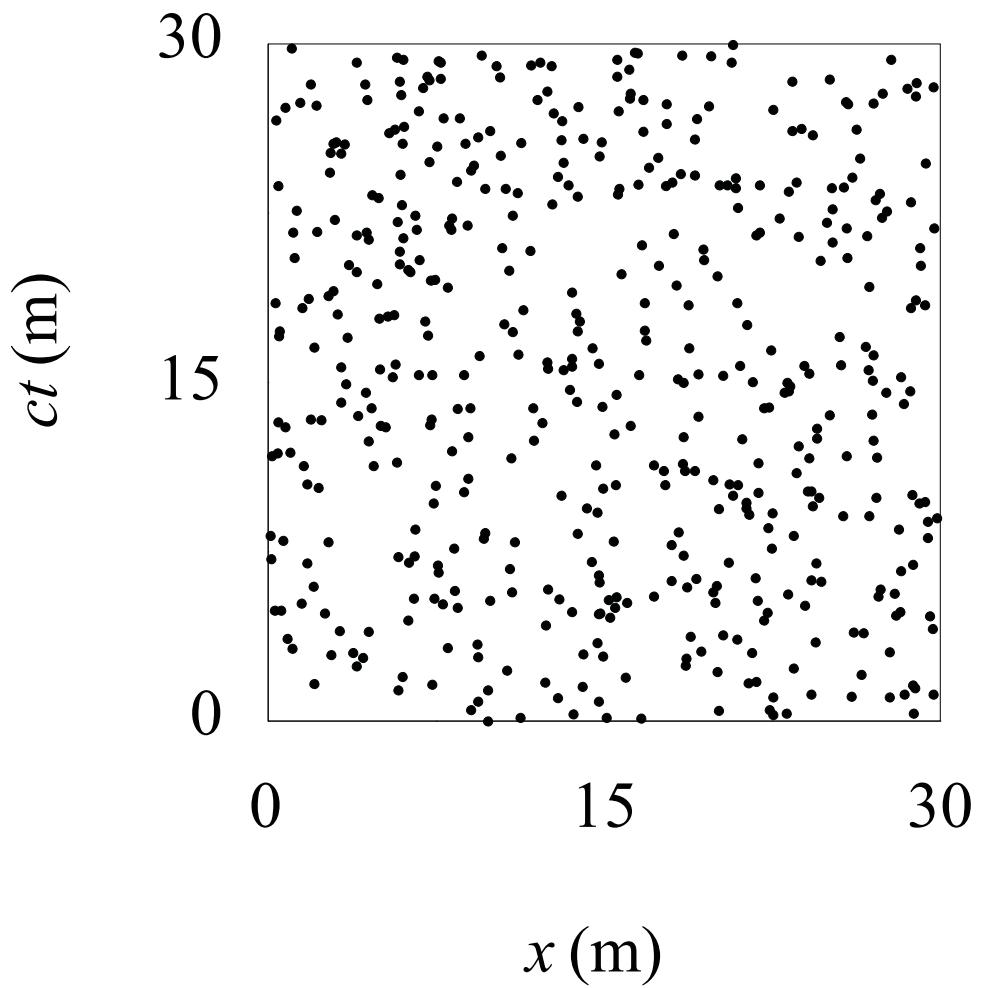


Figure 1

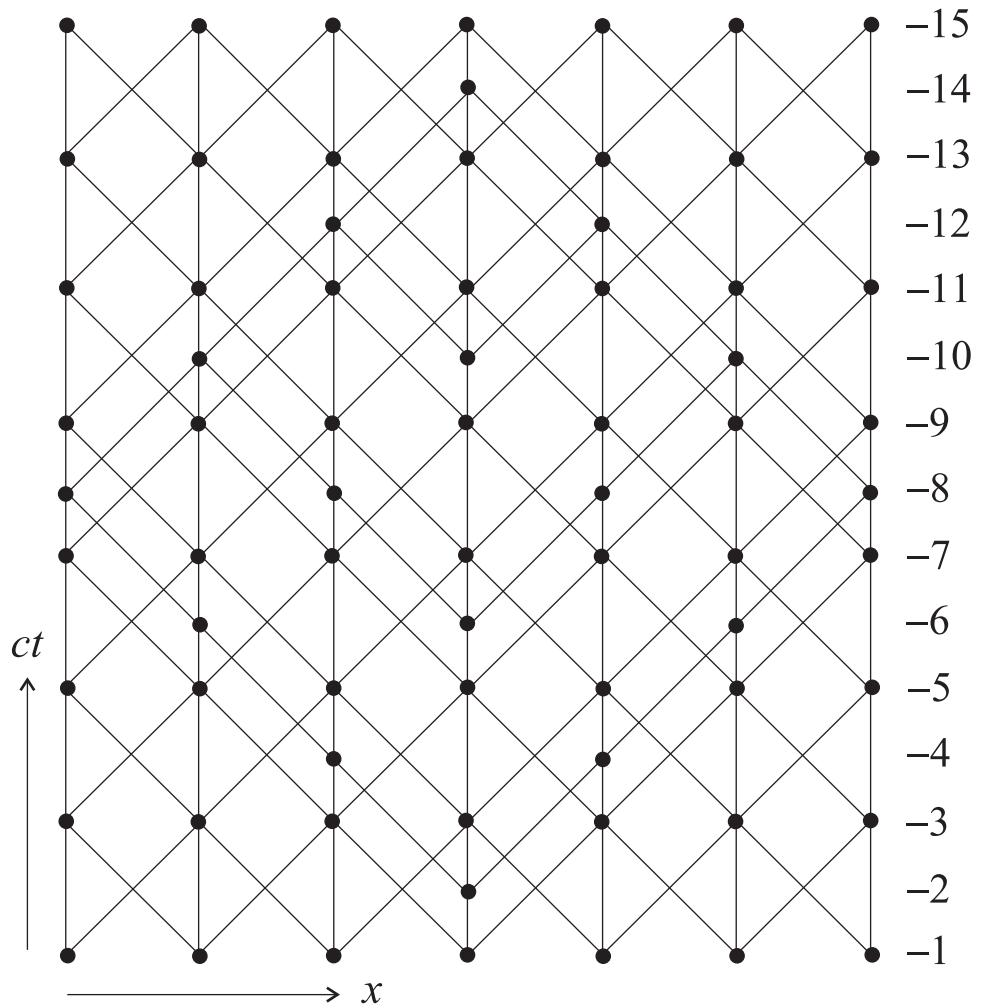


Figure 2

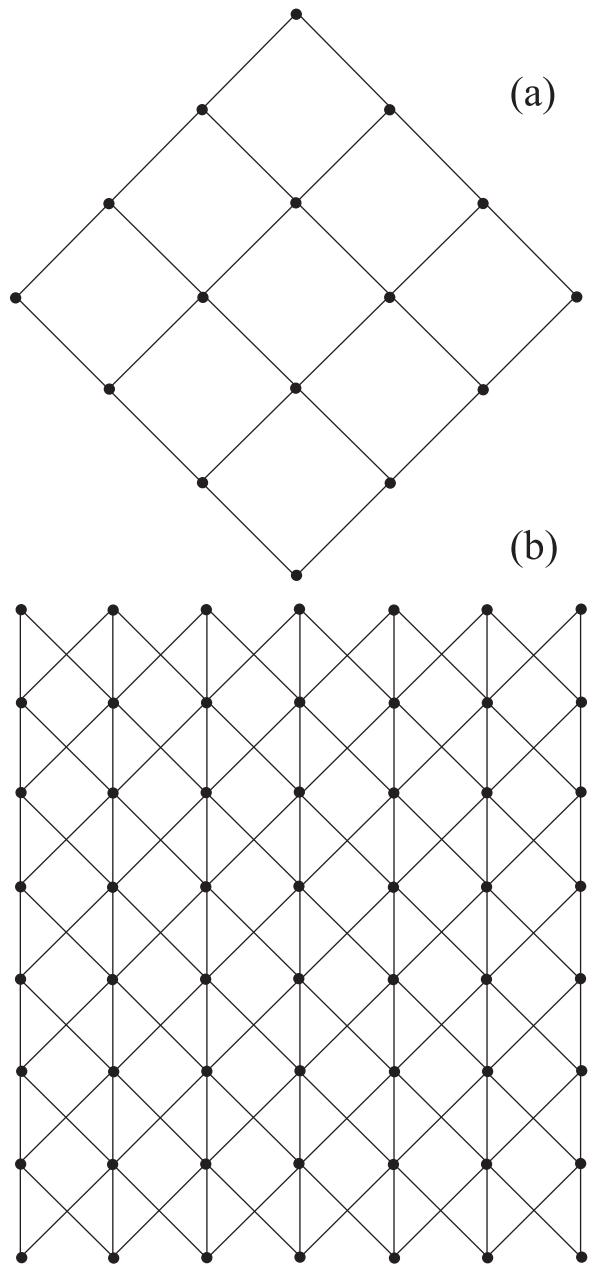


Figure 3